Drawing the Horton Set in an Integer Grid of Minimum Size

Luis Barba *† Frank Duque ‡ Ruy Fabila-Monroy ‡
Carlos Hidalgo-Toscano ‡

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Abstract

In 1978 Erdős asked if every sufficiently large set of points in general position in the plane contains the vertices of a convex k-gon, with the additional property that no other point of the set lies in its interior. Shortly after, Horton provided a construction—which is now called the Horton set—with no such 7-gon. In this paper we show that the Horton set of n points can be realized with integer coordinates of absolute value at most $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$. We also show that any set of points with integer coordinates combinatorially equivalent (with the same order type) to the Horton set, contains a point with a coordinate of absolute value at least $c \cdot n^{\frac{1}{24}\log(n/2)}$, where c is a positive constant.

1 Introduction

Although the number of distinct sets of n points in the plane is infinite, for most problems in Combinatorial Geometry only a finite number of them can be considered as essentially distinct. Various equivalence relations on point sets have been proposed by Goodman and Pollack [12, 14, 15, 13]. One of these is the *order type*; it is defined on a set of points, S, as follows. To every triple (p, q, r) of points of S, assign: a -1 if r lies to the left of the oriented line from p to q; a 1 if r lies to the right of this line; and a 0 if p, q, and r are collinear. This assignment is also called the orientation of the triple, which may be *negative*, *positive* or *zero*, respectively. The set of all triples together with their orientation is the order type of S; two sets of points have the same order type if there is a bijection between them that preserves orientations.

Since their inception, order types were defined with computational applications in mind (see [13] for example). The orientation of a triple is determined

*Département d'Informatique, ULB, Belgium.
†School of Computer Science, Carleton University, Canada.
‡Departamento de Matemáticas, CINVESTAV. Partially supported by Conacyt of Mexico, Grant 153984.

by the sign of a determinant; many algorithms use precisely this determinant as their geometric primitive. Given that the determinant of an integer valued matrix is an integer, for numerical computations it is best if a point set has integer coordinates. Two main reasons are that integer arithmetic is much faster than floating point arithmetic, and that floating point arithmetic is prone to rounding errors. The latter is easily taken care of with an integer representation that can handle arbitrarily large numbers.

If a set of n points has already integer coordinates, it is best if these coordinates have as small absolute value as possible—again, for computational reasons. Even though rounding errors can be avoided using arbitrarily large integers, the cost of computation increases as the numbers get larger. Also, if we wish to store the point set, the number of bits needed depend on the size of the coordinates.

Let S be a set of n points in general position in the plane. A *drawing* of S is a set of points with integer coordinates and with the same order type as S. The *size* of a drawing is the maximum of the absolute values of its coordinates. For the reasons mentioned above, it is of interest to find the drawing of S of minimum size. In [16] Goodman, Pollack and Sturmfels presented sets of n points in general position whose smallest drawings have size $2^{2^{c_1 n}}$, and proved that every point set has a drawing of size at most $2^{2^{c_2 n}}$ (where c_1 and c_2 are positive constants).

Aichholzer, Aurenhammer and Krasser [1] have assembled a database of drawings. For $n = 3, \ldots, 11$, the database contains a drawing of every possible set of n points in general position in the plane. The main advantage of having these drawings is that one can use them to compute certain combinatorial parameters of all point sets up to eleven points. The order type data base stops at eleven because the size of the database grows prohibitively fast. Thus, we cannot hope to store drawings for all point sets beyond small values of n; it is convenient however, to have programs that generate small drawings of infinite families of point sets which are of known interest in Combinatorial Geometry.

In this direction, Bereg et al. [6] provided a linear time algorithm to generate a drawing of a set of points called the Double Circle[1]. Their drawing has size $O(n^{3/2})$; they also proved a lower bound of $\Omega(n^{3/2})$ on the size of every drawing of the Double Circle. In this paper we do likewise for a point set called the Horton Set [18]. In section 2 we provide a drawing of size $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$ of the Horton set of n points; our drawing can be easily constructed in linear time. We also show in Section 3, a lower bound of $c \cdot n^{\frac{1}{24}\log(n/2)}$ (for some $c > 0$) on the minimum size of any drawing of the Horton set. As a corollary, $\Theta(n \log^2 n)$ bits are necessary and sufficient to store a drawing of the Horton set.

We are mainly interested in having an algorithm that generates small drawings of the Horton set. However, the problem of finding small drawings also raises interesting theoretical questions. For example, after learning of our lower bound, Alfredo Hubard posed the following problem.

[1] The Double Circle of $2n$ points is constructed as follows. Start with a convex n-gon; arbitrarily close to the midpoint of each edge, place a point in the interior of this polygon; finally place a point at each vertex of the polygon.

Problem 1. *Does every sufficiently large set of points, for which there exist a drawing of polynomial size, contains an empty* 7*-hole?*

A *k-hole* of a point set S, is a subset of k points of S that form a convex polygon, with no other point of S in its interior. Horton sets were constructed as an example of arbitrarily large point sets without 7-holes. In particular our lower bound implies that any set of points that has a drawing of polynomial size, cannot have large copies of the Horton set. We also believe that the machinery developed to prove Theorem 3.6 will be useful for analyzing Horton sets in other settings.

A preliminary version of this paper appeared in CCCG'14 [5]. In this paper all point sets are in general position and all logarithms are base 2.

1.1 The Horton Set(s)

In 1978 Erdős [9] asked if for every $k \geq 3$, any sufficiently large set of points in the plane contains a k-hole. Shortly after, Harborth [17] showed that every set of 10 points contains a 5-hole. The case of empty triangles (3-holes) is trivial; the case of 4-holes was settled in the affirmative in another context by Esther Klein long before Erdős posed his question (see [10]). Horton [18] constructed arbitrarily large point sets without 7-holes, and thus without k-holes for larger values of k. His construction is now known as the Horton set. The case of 6-holes remained open for almost 30 years, until Nicolás [21], and independently Gerken [11], proved that every sufficiently large set of points contains a 6-hole.

Since its introduction, the Horton set has been used as an extremal example in various combinatorial problems on point sets. For example, a natural question is to ask: What is the minimum number of k-holes in every set of n points in the plane? The case of empty triangles was first considered by Katchalski and Meir [19]—they constructed a set of n points with $200n^2$ empty triangles and showed that every set of n points contains $\Omega(n^2)$ of them. This bound was later improved by Bárány and Füredi [3], who showed that the Horton set has $2n^2$ empty triangles. The Horton set was then used in a series of papers as a building block to construct sets with fewer and fewer k-holes. The first of these constructions was given by Valtr [24]; it was later improved by Dumitrescu [8] and the final improvement was given by Bárány and Valtr [4].

Devillers et al. [7] considered chromatic variants of these problems. In particular, they described a three-coloring of the points of the Horton set with no empty monochromatic triangles. Since every set of 10 points contains a 5-hole, every two-colored set of at least 10 points contains an empty monochromatic triangle. The first non trivial lower bound of $\Omega(n^{5/4})$, on the number of empty monochromatic triangles in every two-colored set of n points, was given by Aichholzer et al [2]. This was later improved by Pach and Tóth [22] to $\Omega(n^{4/3})$. The known set with the least number of empty monochromatic triangles is given in [2]; it is based on the known set with the fewest number of empty triangles, which in turn is based on the Horton set.

We now define the Horton set. Let S be a set of n points in the plane with no

two points having the same x-coordinate; sort its points by their x-coordinate so that $S := \{p_0, p_1, \dots, p_{n-1}\}$. Let S_{even} be the subset of the even-indexed points, and S_{odd} be the subset of the odd-indexed points. That is, $S_{\text{even}} = \{p_0, p_2, \dots\}$ and $S_{\text{odd}} = \{p_1, p_3, \dots\}$. Let X and Y be two sets of points in the plane. We say that X is *high above* Y if: every line determined by two points in X is above every point in Y, and every line determined by two points in Y is below every point in X.

Definition 1. *The **Horton set** is a set H^k of 2^k points, with no two points having the same x-coordinate, that satisfies the following properties.*

1. *H^0 is a Horton set;*
2. *both H^k_{even} and H^k_{odd} are Horton sets ($k \geq 1$);*
3. *H^k_{odd} is high above H^k_{even} ($k \geq 1$).*

This definition is very similar to the one given in Matoušek's book [20] (page 36). The only difference is that in that definition either H^k_{even} is high above H^k_{odd} or H^k_{odd} is high above H^k_{even}; i.e. this relationship is allowed to change at each step of the recursion. As a result, for a fixed value of k, one gets a family of "Horton sets" (with different order types), rather than a single Horton set. Normally, this does not affect the properties that make Horton sets notable. For example, none of the them have empty heptagons. In some circumstances it does; as is the case of the constructions with few k-holes [24, 8, 4]. We fixed one of these two options in order to make the proof of our lower bound more readable, but our results should hold for the general setting. Note that this choice fixes the order type of the Horton set. However, an arbitrary drawing of the Horton set need not satisfy Definition 1.

Horton described his set in a concrete manner with specific integer coordinates. Another description given in [3], is the following.

- $H^0 := \{(1, 1)\}$.
- $H^1 := \{(1, 1), (2, 2)\}$.
- $H^k := \{(2x - 1, y) : (x, y) \in H^{k-1}\} \cup \{(2x, y + 3^{2^{k-1}}) : (x, y) \in H^{k-1}\}$.

This drawing and the original due to Horton have exponential size; we have not seen in the literature a drawing of subexponential size. Then again, to the best of our knowledge nobody has tried to find small drawings of the Horton set.

2 Upper bound

In this section we construct a small drawing of the Horton set of $n := 2^k$ points. First, we define the following two functions.

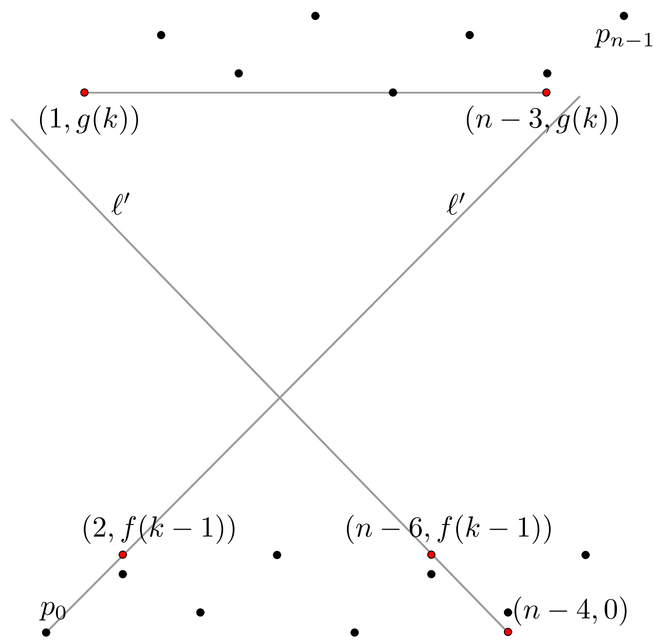

Figure 1: The two possible definitions of ℓ' in the proof of Theorem 2.1.

$$f(i) = \begin{cases} 0 & \text{if } i = 1. \\ 2^{\frac{i(i-1)}{2}-1} & \text{if } i \geq 2. \end{cases}$$

$$g(i) = \begin{cases} 0 & \text{if } i = 1. \\ f(i) - f(i-1) & \text{if } i \geq 2. \end{cases}$$

Afterwards, we use f and g to construct our drawing recursively as follows.

- $P^0 := \{(0,0)\}$;
- $P^i_{\text{even}} := \{(2x, y) : (x, y) \in P^{i-1}\}$;
- $P^i_{\text{odd}} := \{(2x+1, y+g(i)) : (x, y) \in P^{i-1}\}$;
- $P^i := P^i_{\text{even}} \cup P^i_{\text{odd}}$.

Theorem 2.1. *There exist a drawing of the Horton set of n points of size $\frac{1}{2} n^{\frac{1}{2}\log(n/2)}$ for $n \geq 16$.*

Proof. We prove by induction on k that P^k is the desired drawing. It can be verified by hand that P^4 has size equal to $32 = \frac{1}{2} 16^{\frac{1}{2}\log(16/2)}$; assume that

$k \geq 5$. By induction P^k_{even} and P^k_{odd} are Horton sets; it only remains to show that P^k_{odd} is high above P^k_{even}. We only prove that every point of P^k_{odd} is above every line through two points of P^k_{even}; the proof that every point of P^k_{even} is below every line through two points of P^k_{odd} is analogous.

Let $p_0, p_1, \ldots, p_{n-1}$ be the points of P^k sorted by their x-coordinate. Let $0 \leq i < j \leq n-1$ be two even integers, and let ℓ be the directed line from p_i to p_j. By definition P^k_{odd} is above the vertical line passing through p_1; in particular $P^k_{\text{odd}} \setminus \{p_{n-1}\}$ is above the line segment joining p_1 and p_{n-3}. Since the smallest y-coordinate of P^k_{odd} is equal to $g(k)$, p_1 and p_{n-3} are above the line segment joining the points $(1, g(k))$ and $(n-3, g(k))$. Therefore, it suffices to show that $(1, g(k))$, $(n-3, g(k))$ and p_{n-1} are above ℓ.

We define a line ℓ' with the property that if $(1, g(k))$ and $(n-3, g(k))$ are above ℓ', then $(1, g(k))$, $(n-3, g(k))$ and p_{n-1} are above ℓ. Afterwards we show that indeed $(1, g(k))$ and $(n-3, g(k))$ are above ℓ'.

If the slope of ℓ is non-positive, define ℓ' to be the line passing through the points $(n-6, f(k-1))$ and $(n-4, 0)$; if the slope of ℓ is positive, define ℓ' to be the line passing through the points $(0,0)$ and $(2, f(k-1))$. Note that the largest y-coordinate of P^k_{even} is equal to $\sum_{i=1}^{k-1} g(i) = f(k-1)$. Therefore the slope of ℓ is at least $-f(k-1)/2$ and at most $f(k-1)/2$; in particular the absolute value of the slope of ℓ' is larger or equal to the absolute value of the slope of ℓ. The farthest point of P^k_{even} to the right that ℓ can contain while having non-positive slope is p_{n-4} (which has x-coordinate equal to $n-4$); the farthest point of P^k_{even} to the left that ℓ can contain while having positive slope is p_0. Therefore in both cases if $(1, g(k))$ and $(n-3, g(k))$ are above ℓ', then they are also above ℓ; see Figure 1.

If ℓ has non-positive slope and $(1, g(k))$ is above ℓ, then p_{n-1} is also above ℓ' since p_{n-1} has larger x-coordinate. If ℓ has positive slope and $(n-3, g(k))$ is above ℓ', then p_{n-1} must also be above ℓ. Otherwise ℓ intersects the line segment joining $(n-3, g(k))$ and p_{n-1}; this line segment has slope equal to $f(k-1)/2$, since the y-coordinate of p_{n-1} is equal to $\sum_{i=1}^{k} g(i) = f(k)$. This in turn would imply that ℓ has slope larger than $f(k-1)/2$—a contradiction.

Suppose ℓ has non-positive slope. Then it suffices to show that $(1, g(k))$ is above ℓ'. This is the case since:

$$
\begin{aligned}
& \begin{vmatrix} n-6 & f(k-1) & 1 \\ n-4 & 0 & 1 \\ 1 & g(k) & 1 \end{vmatrix} \\
= \;& 2g(k) - (n-5)f(k-1) \\
= \;& 2f(k) - (n-3)f(k-1) \\
= \;& 2f(k) - 2^k f(k-1) + 3f(k-1) \\
= \;& 3f(k-1) \\
> \;& 0.
\end{aligned}
$$

Suppose ℓ has positive slope. Then it suffices to show that $(n-3, g(k))$ is

above ℓ'. This is the case since:

$$\begin{aligned}
&\begin{vmatrix} 0 & 0 & 1 \\ 2 & f(k-1) & 1 \\ n-3 & g(k) & 1 \end{vmatrix} \\
=&\ 2g(k) - (n-3)f(k-1) \\
=&\ 2f(k) - (n-1)f(k-1) \\
=&\ 2f(k) - 2^k f(k-1) + f(k-1) \\
=&\ f(k-1) \\
>&\ 0.
\end{aligned}$$

Finally the largest x-coordinate of P^k is equal to $n-1$, and the largest y-coordinate of P^k is equal to

$$\sum_{i=1}^{k} g(i) = f(k) = 2^{\frac{k(k-1)}{2}-1} = \frac{1}{2} n^{\frac{1}{2}\log(n/2)},$$

since $k = \log n$. Therefore, P^k is a drawing of the Horton set of n points of size $\frac{1}{2} n^{\frac{1}{2}\log(n/2)}$. $\square$

3 Lower bound

In this section we prove a lower bound on the size of any drawing of the Horton set. As mentioned before, a drawing of the Horton set might not satisfy Definition 1; we call a drawing that does, an *isothetic* drawing of the Horton set. We first show a lower bound on the size of isothetic drawings of the Horton set (Theorem 3.5); afterwards, we consider the general case (Theorem 3.6). Throughout this section P is an isothetic drawing of the Horton set of $n := 2^k$ points, and $p_0, p_1, \ldots, p_{n-1}$ are the points of P sorted by their x-coordinate.

As an auxiliary structure, we recursively define a complete rooted binary tree T, as follows. P is the root of T; and if $Q \subset P$ is a vertex of T, of at least two points, then Q_{even} and Q_{odd} are its left and right children, respectively. Furthermore, for each vertex in T, label the edge incident to its left child with a "0" and the edge incident with its right child with a "1"; the labels encountered in a path from a leaf $\{p_i\}$ to the root are precisely the bits in the binary expansion of i; see Figure 2.

By construction, the vertices of T are sets of 2^i points of P (for some $0 \leq i \leq k$). Let T_i be the set of vertices of T that consist of exactly 2^i points of P: we call it the i-th[2] level of T. The first level, T_1, are the vertices of T that consist of a pair of points of P. For each such pair, we consider the line through them as defined by them.

[2]In the literature the i-th level of a binary tree are those vertices at distance i from the root; we have precisely the opposite order.

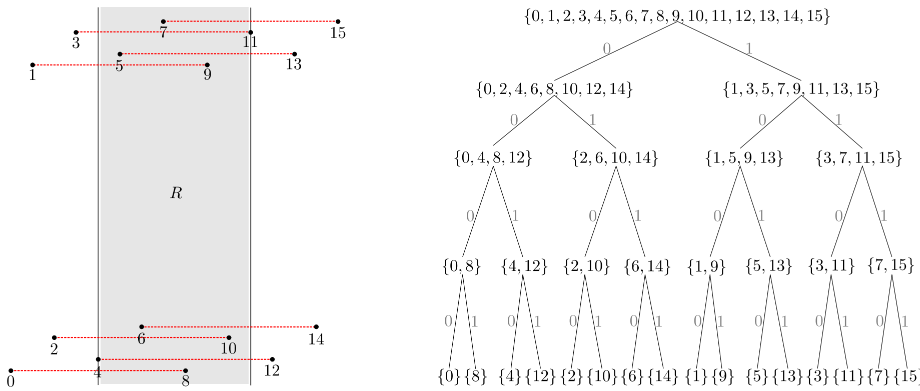

Figure 2: The Horton set and its associated tree T.

Let R be the closed vertical slab bounded by the vertical lines through $p_{n/4}$ and $p_{3n/4-1}$. Let Q be a vertex at the first level of T and let p_i and p_j be its leftmost and rightmost points respectively. Suppose that Q is a left child. Then the two most significant bits in the binary expansion of i are "00", and the two most significant bits in the binary expansion of j are "10". This implies that $i \leq n/4$ and $j - i = n/2$; in particular p_j is contained in R, while p_i is to the left of R. In this case, we say that Q is *left-to-right* crossing. By similar arguments if Q is a right child, then p_i is contained in R, while p_j is to the right of R. In this case we say that Q is *right-to-left* crossing. Note that the vertices in the first level of T, in their left to right order in T, are alternatively *left-to-right* and *right-to-left* crossing (see Figure 2). The following lemma relates the left to right order of these vertices in T, to the bottom-up order of their corresponding pairs of points of P.

Lemma 3.1. *The lines defined by the vertices of the first level of T do not intersect inside R. In particular, the bottom-up order of convex hull of these vertices corresponds to their left to right order in T.*

Proof. Let Q_1 and Q_2 be two vertices in the first level of T such that Q_1 is a left child and Q_2 is a right child. Without loss of generality assume that in the left to right order in T, Q_1 is before Q_2. Then, Q_1 is left-to-right crossing and Q_2 is right-to-left crossing. Let γ_1 and γ_2 be the lines defined by Q_1 and Q_2, respectively. If γ_1 and γ_2 intersect inside R, then the leftmost point of Q_1 is above γ_2 or the rightmost point of Q_2 is below γ_1—a contradiction to property 3 of Definition 1. Since between every two left children there is a right child, and between every two right children there is a left child, the result follows. $\square$

By construction every vertex of T is an isothetic drawing of the Horton set. The main idea behind the proof of the lower bound on the size of isothetic drawings of the Horton set is to lower bound the size of these drawings in terms

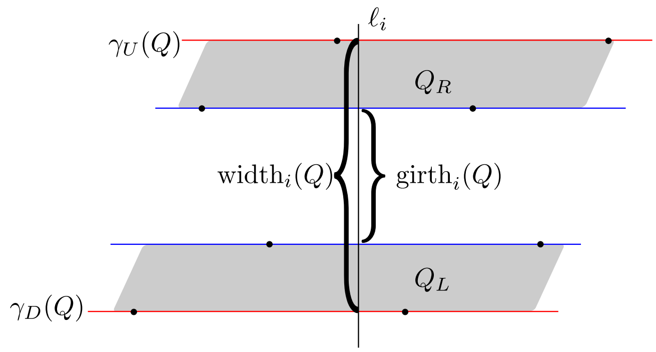

Figure 3: The bounding lines of Q, together with its width and girth.

of the size of their children. We define some parameters on the vertices of T, that make this idea more precise.

Let $2 \leq t \leq k$ be an integer. Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four vertical lines sorted from left to right, such that:

- All of them are contained in the interior of R.
- There are exactly 2^{k-t} points of P between both pairs (ℓ_1, ℓ_2) and (ℓ_3, ℓ_4).

Let Q be a vertex of T with more than two points. For each of the ℓ_i, we define two parameters of Q. Let $\gamma_D(Q)$ be the line defined by the leftmost descendant of Q in T_1. Let $\gamma_U(Q)$ be the line defined by the rightmost descendant of Q in T_1. Note that Q is bounded from below by $\gamma_D(Q)$ and from above by $\gamma_U(Q)$ (Lemma 3.1). Let Q_L and Q_R be the left and right children of Q, respectively. Define $\text{width}_i(Q)$ as the distance between the points $\gamma_D(Q) \cap \ell_i$ and $\gamma_U(Q) \cap \ell_i$, and $\text{girth}_i(Q)$ as the distance between the points $\gamma_U(Q_L) \cap \ell_i$ and $\gamma_D(Q_R) \cap \ell_i$; see Figure 3.

We lower bound the girth of a vertex of T in terms of the girth of one of its children. This bound is expressed in Lemma 3.2. Before proceeding we need one more definition. Let Q be a vertex of T with more than two points and let $P(Q)$ be its parent. If Q is the left child of $P(Q)$, let $S(Q)$ be the right child of Q; otherwise let $S(Q)$ be the left child of Q.

Lemma 3.2. *Let Q be a vertex at the l-th level of T, for some $t < l < k$. If the distance between ℓ_1 and ℓ_2 is d_1, and the distance between ℓ_3 and ℓ_4 is d_2, then:*

(1) $\text{girth}_1(P(Q)) \geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{l-t-1}\, \text{girth}_4(Q) - \text{width}_1(S(Q))$ *and,*

(2) $\text{girth}_4(P(Q)) \geq \left(\frac{(d_2)^2}{(d_1+d_2)d_1}\right) 2^{l-t-1}\, \text{girth}_1(Q) - \text{width}_4(S(Q))$.

Proof. We will prove inequality (1); the proof of (2) is analogous. Assume that Q is the left child of $P(Q)$ and let Q' be the right child of $P(Q)$; the case when

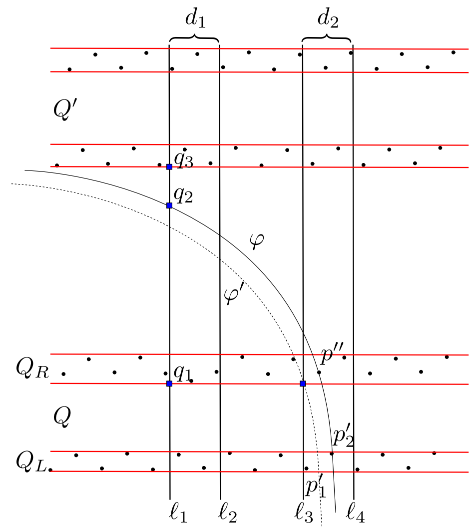

Figure 4: Schematic depiction of the proof of Lemma 3.2.

Q is the right child of $P(Q)$ can be proven with similar arguments. Note that is $S(Q)$ is the right child, Q_R, of Q.

Let p_1' and p_2' be two consecutive points in Q_L lying between ℓ_3 and ℓ_4 at a horizontal distance of at most $\Delta_x := d_2/2^{l-t-1}$ from each other; such a pair exists as there are 2^{l-t-1} points of Q_L between ℓ_3 and ℓ_4. Let p'' be the point in Q_R that lies between p_1' and p_2' (in the x-coordinate order). Let φ be the line through p_2' and p''. Let $\Delta_y := \min\{\text{girth}_3(Q), \text{girth}_4(Q)\}$; note that the slope of φ is at most $-\Delta_y/\Delta_x$. Recall that by Lemma 3.1, $\gamma_D(Q_R)$ and $\gamma_U(Q_L)$ do not intersect between ℓ_1 and ℓ_4; this implies that $\text{girth}_3(Q) \geq \frac{d_1}{d_1+d_2}\,\text{girth}_4(Q)$, in particular $\Delta_y \geq \frac{d_1}{d_1+d_2}\,\text{girth}_4(Q)$. Therefore, the slope of φ is at most

$$-\Delta_y/\Delta_x = -\left(\frac{d_1}{d_1+d_2}\,\text{girth}_4(Q)\right)/\Delta_x = \frac{d_1}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_4(Q).$$

Define the following points $q_1 := \gamma_D(Q_R) \cap \ell_1$, $q_2 := \varphi \cap \ell_1$ and $q_3 := \gamma_D(Q') \cap \ell_1$ (see Figure 4). Note that the leftmost point of $\gamma_D(Q') \cap Q'$ is to the left of ℓ_1; since this point is above φ, q_2 cannot be above q_3. Therefore, the distance from q_1 to q_2 is at most the distance from q_1 to q_3; the distance from q_1 to q_3 is precisely $\text{girth}_1(P(Q)) + \text{width}_1(S(Q))$. We now show that the distance from q_1 to q_2 is at least $\frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_4(Q)$—this completes the proof of (1).

Let φ' be the line parallel to φ and passing through the intersection point of ℓ_3 and $\gamma_D(Q_R)$. Note that φ' is below φ. Therefore, the distance from q_1 to q_2 is at least the distance of q_1 to the intersection point of φ': this is at least $d_1(\Delta_y/\Delta_x) = \frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_4(Q)$. $\square$

Two obstacles prevent us from directly applying Lemma 3.2. One is that the difference between d_1 and d_2 may be too big and in consequence $\frac{(d_1)^2}{(d_1+d_2)d_2}$ or $\frac{(d_2)^2}{(d_1+d_2)d_1}$ too small. This situation can be fixed with following Lemma.

Lemma 3.3. *For $t := \lceil 2\log k \rceil$ and $k \geq 16$, P has size at least $n^{\frac{1}{2}\log n}$ or $\ell_1, \ell_2, \ell_3, \ell_4$ can be chosen so that the ratio between d_1 and d_2 is at least $1/2$ and at most 2.*

Proof. Let $\varphi_1, \dots, \varphi_{2^{t-1}}$ be consecutive vertical lines such that:

- all of them lie in the interior of R and,
- between every pair of two consecutive lines $(\varphi_i, \varphi_{i+1})$ there are exactly 2^{k-t} points of P.

For $1 \leq i < 2^{t-1}$, let Δ_i be the distance between φ_i and φ_{i+1}; let $\Delta'_1 \leq \Delta'_2 \leq \dots \leq \Delta'_{2^{t-1}-1}$ be these distances sorted by size. We look for a pair $\Delta'_i \leq \Delta'_j$, such that one is at most two times the other. Suppose there is no such pair; then $\Delta'_{i+1} \geq 2\Delta'_i$. Since between the two lines defining Δ'_1 there are exactly 2^{k-t} points of P, and no three of them have the same integer x-coordinate, $\Delta'_1 \geq 2^{k-t-1}$. Therefore,

$$\Delta'_{2^{t-1}-1} \geq 2^{k-t-1} \cdot 2^{2^{t-1}-2} \geq 2^{\frac{1}{2}k^2+k-t-3} \geq 2^{\frac{1}{2}k^2+k-2\log(k)-4} \geq n^{\frac{1}{2}\log n}.$$

The latter part of the inequality follows from our assumption that $k \geq 16$. Therefore, if there is no such pair, P has size at least $n^{\frac{1}{2}\log n}$.

□

The second obstacle is that the second term in the right hand sides of inequalities (1) and (2) of Lemma 3.2 may be too large. In this case, we prune T to get rid of vertices of large width; this is done by choosing an integer $l \leq k-1$ and then removing from P all the points are contained in either: all the vertices of T_l that are a left child to their parent, or all the the vertices of T_l that are a right child to their parent (see Figure 5). We call this operation *pruning* the l-th level of T. The resulting set is a drawing of the Horton set, as shown by the following lemma.

Lemma 3.4. *Let P' be the subset of P that results from pruning the l-th level of T. Then:*

(1) P' is an isothetic drawing of the Horton set of $n/2$ points.

(2) Suppose that $l \leq k-3$. Let T' be the tree associated to P', and Q' be any vertex at the l-th level of T' (for some $l' > l$). Then there exist a vertex Q at the $(l'+1)$-level of T that contains Q'. Moreover, $S(Q') \subset S(Q)$.

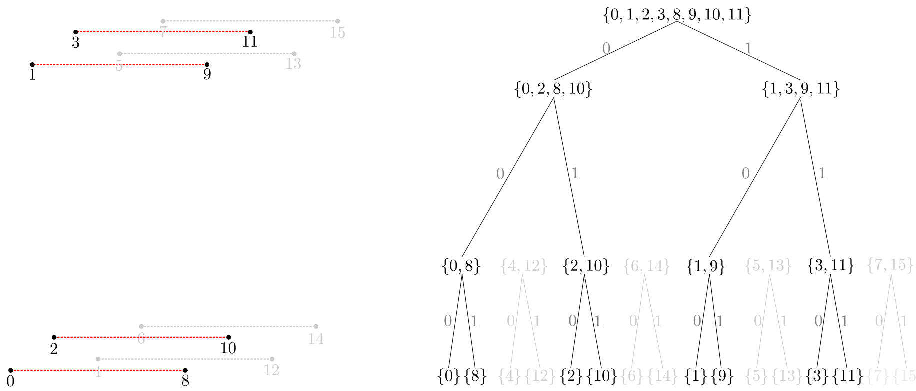

Figure 5: P and T after the removal of the vertices of T_1 that are a right child to their parent.

Proof. Assume without loss of generality that the left children are removed when pruning T. If $l = k-1$, (2) holds trivially, and (1) holds because in that case $P' = P_{\text{odd}}$. Assume that $l \leq k-2$, and let $s := k-l$; we proceed by induction on s.

Note that P_{even} and P_{odd} are each an isothetic drawing of the Horton set of $n/2$ points. Moreover, their corresponding trees, T_{even} and T_{odd}, are the subtrees of T rooted at P_{even} and P_{odd}, respectively. Therefore, when we prune the l-th level of T, we also prune the l-th level of T_{even} and T_{odd}. By induction and (1), this produces two isothetic drawings of the Horton set of $n/4$ points; let $P_0'' \subset P_{\text{even}}$ and $P_1'' \subset P_{\text{odd}}$ be these drawings, respectively.

We first prove that

P' can be constructed from P by, starting at p_0, alternatively removing and keeping intervals of 2^{k-l-1} consecutive points of P. $(*)$

For $s = 1$, this is trivial since $2^{k-l-1} = 1$ and $P' = P_{\text{odd}}$. Thus, by induction, P_0'' and P_1'' are constructed from P_{even} and P_{odd} by, starting at their leftmost point, alternatively removing and keeping intervals of 2^{k-l-2} consecutive points of P_{even} and P_{odd}, respectively. Let $I_1', \ldots, I_{2^{l+1}}' \subset P_{\text{even}}$ and $J_1', \ldots, J_{2^{l+1}}' \subset P_{\text{odd}}$ be these intervals (in order). Finally, $(*)$ follows by letting $I_i := I_i' \cup J_i'$.

We now prove *1* and *2*.

(1) Note that $(*)$ implies that $P_{\text{even}}' = P_0''$ and $P_{\text{odd}}' = P_1''$. Thus $P_{\text{even}}' \subset P_{\text{even}}$ and $P_{\text{odd}}' \subset P_{\text{odd}}$; in particular P_{odd}' is high above P_{even}'. Therefore, P' is an isothetic drawing of the Horton set of $n/2$ points.

(2) Consider the following algorithm. Remove from T the subtrees rooted at the vertices in the l-th level of T that are a left child to their parent;

afterwards, remove from each vertex of T the points in $P \setminus P'$. After this last step, each vertex at the l-th level of T that was not removed is equal to its parent—producing a loop; remove these loops. We claim that this algorithm produces T'. For $s = 1$, this follows from $(*)$. Let T'_{even} and T'_{odd} be the left and right subtrees of the root of T', respectively. By induction T'_{even} and T'_{odd} can constructed from T_{even} and T_{odd} with the above algorithm, respectively. Since the root of T' is precisely the root of T minus the points in $P \setminus P'$, the algorithm produces T'.

Now, suppose that $l \leq k-3$ and let Q' be a vertex at the l'-th level of T', for some $l' > l$. By the algorithm, there is a vertex Q such that $Q' = Q \setminus (P \setminus P')$; this vertex is in the $(l'+1)$-level of T. Finally, also by the algorithm we have that $S(Q') \subset S(Q)$.

□

We are now ready to prove our lower bound on the size of isothetic drawings of the Horton set.

Theorem 3.5. *For a sufficiently large value of k, every isothetic drawing of the Horton set of $n = 2^k$ points has size at least $n^{\frac{1}{8}\log n}$.*

Proof. Set $t := \lceil 2 \log k \rceil$ and assume that $k \geq 16$. By Lemma 3.3 ℓ_1, ℓ_2, ℓ_3 and ℓ_4 can be chosen so that, the ratio of the distance between d_1 and d_2 is at least $1/2$ and at most 2. Without loss of generality assume that $d_1 \leq d_2$. Let D be the distance between ℓ_1 and ℓ_4. We may assume that

$$D < n^{\frac{1}{8}\log n};$$

as otherwise we are done.

Let Q be a vertex in the $(t+1)$-th level of T. Note that between two consecutive points in every vertex at the l-th level of T there are exactly $2^{k-l}-1$ points of P. This trivially holds for $l = k$; it holds for smaller values of l, by induction on $k-l$. In particular, there are $(2^{t+1}-1)(2^{k-t-1}-1) + 2^{t+1} - 2 = 2^k - 2^{k-t-1} - 1$ points of P between the leftmost and rightmost point of Q.

This implies that there are exactly two points of Q between ℓ_1 and ℓ_2, and exactly two points of Q between ℓ_3 and ℓ_4.

Suppose that there were less than two points of Q between ℓ_1 and ℓ_2, then the number of points of P would be at least the sum of the following.

- The number of points of P between ℓ_1 and ℓ_2; recall that this is equal to 2^{k-t}.

- The number of points of P between the leftmost and rightmost point of Q that are not between ℓ_1 and ℓ_2; since there are exactly $2^{k-t-1}-1$ points of P between two consecutive points of Q, and at most one point of Q between ℓ_1 and ℓ_2, this is at least $(2^k - 2^{k-t-1} - 1) - 2^{k-t-1} = 2^k - 2^{k-t}$.

- Two, for the leftmost and rightmost point of Q.

In total this is at least $2^k + 1 = n + 1$—a contradiction; similar arguments hold for ℓ_3 and ℓ_4.

Suppose that there are more than two points of Q between ℓ_1 and ℓ_2, then the number of points of P between ℓ_1 and ℓ_2 is at least $2(2^{k-t-1})+3 = 2^{k-t}+3$; this is a contradiction to the assumption that there are exactly 2^{k-t} points of P between ℓ_1 and ℓ_2. The same argument holds for ℓ_3 and ℓ_4.

The two points of Q between ℓ_1 and ℓ_2, and the two points of Q between ℓ_3 and ℓ_4, have integer coordinates. Therefore, by Pick's theorem [23] the area of their convex hull is at least one. Since these points are contained in trapezoid bounded by $\gamma_D(Q)$, $\gamma_U(Q)$, ℓ_1 and ℓ_4, the area of this trapezoid is also at least one. But this area is at most $D(\mathrm{width}_1(Q) + \mathrm{width}_4(Q))/2$. Therefore

$$\max\{\mathrm{width}_1(Q), \mathrm{width}_4(Q)\} \geq 1/D. \tag{1}$$

This bound also holds for every vertex at a level higher than $t+1$ (since all of these vertices contain vertices at the $t+1$-level as subsets).

Let $t < l \leq k$ be the largest positive integer such that there exists a vertex R in the l-th level of T that satisfies:

$$\max\{\mathrm{width}_1(S(R)), \mathrm{width}_4(S(R))\} \geq \frac{2^{(l-t-6)(l-t-7)/2}}{D}. \tag{2}$$

Such an l and R exist since (2) holds for every vertex at the $(t+6)$-th level of T. Indeed if Q is a vertex at the $(t+6)$-th level of T, then $S(Q)$ is in the $(t+5)$ level of T and by (1):

$$\max\{\mathrm{width}_1(S(Q)), \mathrm{width}_4(S(Q))\} \geq 1/D = \frac{2^{((t+6)-t-6)((t+6)-t-7)/2}}{D}.$$

Without loss of generality assume that $\mathrm{width}_1(S(R)) \geq (2^{(l-t-6)(l-t-7)/2})/D$ and that R is a left child. We may assume that $l < k$, otherwise (2) implies that P has size at least $n^{\frac{1}{8}\log n}$ (for a sufficiently large value of k).

We now apply Lemma 3.4 to prune T of all the vertices of large width (that, is that satisfy (2)). Prune the l-th level of T by removing all the vertices that are a left child to their parent. Let P' be the resulting point set and T' its corresponding tree. No vertex of T' in a level higher than l satisfies (2); otherwise, by part (2) of Lemma 3.4 there would be a vertex at level of T higher than l that satisfies (2).

Let $(P(R)' = Q'_l, Q'_{l+1}, \ldots, Q'_{k-1} = P')$ be the path from $P(R)'$ to the root of T'. We prove inductively for $l \leq m \leq k-1$, that:

$$\mathrm{girth}_1(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \equiv l \mod 2, \tag{3}$$

$$\mathrm{girth}_4(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \not\equiv l \mod 2 \tag{4}$$

(3) holds for $m = l$ since $\mathrm{girth}_1(Q'_{l+1}) = \mathrm{girth}_1(P(R)') \geq \mathrm{width}_1(S(R)) \geq (2^{(l-t-6)(l-t-7)/2})/D$. Assume that $m > l$ and that both (3) and (4) hold

for smaller values of m. Suppose that m has the same parity as l. Then by inequality (1) of Lemma 3.2 and inequalities (2) and (3):

$$\begin{aligned}
\operatorname{girth}_1(Q'_m) &\geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{m-t-2} \operatorname{girth}_4(Q'_{m-1}) - \operatorname{width}_1(S(Q'_{m-1})) \\
&\geq 2^{m-t-5} \operatorname{girth}_4(Q'_{m-1}) - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-5} \frac{2^{(m-t-7)(m-t-8)/2}}{D} - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-6} \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&= \frac{2^{(m-t-6)(m-t-7)/2}}{D}
\end{aligned}$$

Therefore P' has size at least $\frac{2^{(k-t-7)(k-t-8)/2}}{D}$. This at least $n^{\frac{1}{8}\log n}$, for a sufficiently large value of k. Since $P' \subset P$, the result follows. The proof when m has different parity as l is similar, but uses inequality (2) of Lemma 3.2 instead. $\square$

To prove the general lower bound we do the following. Take a drawing of the Horton set; find a subset of half of its points, for which we know that there exists a linear transformation that maps it into an isothetic drawing; afterwards, apply Lemma 3.5 to the image and use the obtained lower bound to lower bound the size of original drawing.

Theorem 3.6. *Every drawing of the Horton set of $n = 2^k$ points has size at least $c \cdot n^{\frac{1}{24}\log(n/2)}$, for a sufficiently large value of n and some positive constant c.*

Proof. Let P' be a (not necessarily isothetic) drawing of the Horton set of n points. As P and P' have the same order type we can label P' with the same labels as P, such that corresponding triples of points in P and P' have the same orientation. Let $\{p'_0, \dots, p'_{n-1}\}$ be P' with these labels.

Note that the clockwise order by angle of P'_{odd} around p'_0 is $(p'_1, p'_3, \dots)$, and that p'_0 lies in an unbounded cell of the line arrangement of the lines defined by every pair of points of P'_{odd}; thus, point p'_0 can be moved towards infinity without changing this radial order around p'_0. Therefore, there is a direction $\vec{d}$ in which if P'_{odd} is projected orthogonally the order of the projection is precisely $(p'_1, p'_3, \dots)$. We may rotate $\vec{d}$ as long as it does not coincide with a direction defined by a pair of points of P' and the order of P'_{odd} in this projection does not change. Let v' and v'' be the first vectors, defined by pairs of points of P', encountered when rotating $\vec{d}$ to the left and to the right, respectively; let $v = v' + v'' = (a, b)$.

We may assume that $||v|| = \sqrt{a^2+b^2} \leq 4^{1/3}(n/2)^{\frac{1}{24}\log(n/2)}$; otherwise one of v' and v'' has length at least $(1/2)4^{1/3}(n/2)^{\frac{1}{24}\log(n/2)}$, and therefore a coordinate

of value at least $(1/4)^{2/3}(n/2)^{\frac{1}{24}\log(n/2)}$. Let $v^{\perp} = (b, -a)$. Consider a change of basis from the standard basis to $\{v, v^{\perp}\}$. Note that under this transformation (x, y) is mapped to $\left(\frac{ax+by}{a^2+b^2}, \frac{ay-bx}{a^2+b^2}\right)$. We multiply the image of P' under this mapping by $a^2 + b^2$, to obtain an isothetic drawing of the Horton set on $n/2$ points. By Theorem 3.5, this drawing has size at least $(n/2)^{\frac{1}{8}\log(n/2)}$. Therefore, P' has size at least $((n/2)^{\frac{1}{8}\log(n/2)})/(a^2 + b^2) \geq (1/4)^{2/3}(n/2)^{\frac{1}{24}\log(n/2)}$. $\square$

We point out that the constants in the exponent of the lower bounds of Theorems 3.5 and 3.6 can be improved. We simplified the exposition at the expense of these worse bounds.

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